THE GOLDEN MEAN

WITH

A New Solution for the Parthenon's GM

The Golden Mean is a ratio which has fascinated generation after generation, and culture after culture. It can be expressed succinctly in the ratio of the number "1" to the irrational "l.618034... ", but it has meant so many things to so many people, that a basic investigation of what might is the "Golden Mean Phenomenon" seems in order. So much has been written over the centuries on the Mean, both fanciful imaginings and recondite mathematicizations, that a review of the literature on the subject would be oversize, and probably lose the focus of the problem.

This purpose of this paper is to state in the simplest form problems which relate to the Golden Mean, and pursue a variety of directions which aim to explain the origin of this remarkable ratio and its ultimate meaning in the world of mind and matter.

In modern times there has been much interest in the Golden Proportion, Section or Mean. Since the Renaissance it has been used extensively in art and architecture, it figures in the Venetian Church of St. Mark built early in the 16th century, and has become a standard proportion for width in relation to height as used in facades of buildings, in window sizing, in first story to second story proportion, at times in the dimensions of paintings and picture frames. There is something "satisfactory" about the relationships of the Greek "divided lines" proportion, which some have felt to be modern acculturation since the Renaissance. In the l930's the Pratt Institute of New York did a study on various rectangular proportions laid out as vertical frames, and asked several hundred art students to comment on which seemed the most pleasing. The ratio of 1 : 2 was least liked, while the Golden Ratio was favored by a very large margin, which seemed to point to the actual dimensions as generating a pleasing response by their size.

The French architect LeCorbusier noted that the human body when measured from foot to navel and then again from navel to top of head, showed average numbers very near to the Golden Ratio. He extended this to height compared with arm-span, and designed doorways consonant with these numbers. But of course much of this was based in averages rather than exact numbers, and so falls into the general area of esthetic design, rather than mathematical proportion.

However studies have shown that the patterns of tree- branching adhere to the GM proportion, although again not exactly, while the dendritic cracking in certain metallic alloys which occurs as very low temperatures is basically GM based. In an entirely different area, Duckworth at Princeton found in the early 1940's a GM relationship in the length of paragraphs in Vergil's Aeneid, with the figures becoming ever more accurate as larger samples were taken. Lendvai has demonstrated that Bartok used the GM ratio extensively in composing music, the question remaining whether an artist as an educated person uses the GM ratio consciously as a framework for his work, or unconsciously because of its ubiquitous appearance in the world around us, something we sense by living in a GM proportioned world.

AN ALGEBRAIC APPROACH

FIRST let us examine the Golden Section from a algebraic direction :

The Golden Section is the division of a given unit of length into two parts such that the ratio of the shorter to the longer equals the ratio of the longer part to the whole. Calling the longer part x and accordingly the shorter part 1-x, this condition reads

1-x is to x as x is to 1

$(1-x)/x = x/1$

This is solved by multiplying both sides by x, to get

$1-x = x^2$

or

$x^2 + x - 1 = 0$

The Quadratic Formula (x = (-b +/- sq.r.(b^2 - 4ac))/2a) applies here with a=1, b=1, c=-1, and yields the answer

x = (-1 + sqr(5))/2 =. 618, nearly.

2) SECOND ,I point to the circular method given in standard algebra textbooks, which I can not reproduce here since it demands a diagram and I am using a text-only format for this material. It follows Euclidean procedure in working with a circular display. Briefly, as far back as about 500 BC it was observed that in the regular decagon (figure of 10 equal sides inscribed in a circle), the triangle formed by one of the exterior segments and two radii will show the Golden Proportion in the ratio of short to long leg of that isosceles triangle. Incidentally, its base angles (72 degr.) are just twice its apex angle (36deg). A traditional description of this process in formal terms can be seen in E P Vance's Modern Algebra and Trigonometry, l962. or in any algebra textbook.

This is especially interesting in that it involves the construction of a pentagon and the 10 fold division of a circle, with dimensions which evolve from the 1 : 2 rectangle. The common denominator to both procedures is of course the sqr 5!

Perhaps it is better to see all this in diagram and follow the derivation as given there. This, compared with the previous section, is a somewhat different, non-quadratic way of finding the GM ratio, it is geometric and more in the spirit of the early Greek investigators than the algebraic methods given above.

THIRD: Here is an amateur method of my own, which I present with hesitation, since I have seen no parallel to it elsewhere. Starting with the number one (1), I want to find any number larger than it, the inverse of which is smaller by the difference of one (1) while retaining the same digits. If I try random numbers, I find the difference either too large or too small, so by a rather exhausting session with the Method of Exhaustions, I find my numbers converging on the GM figures:

.618034 and 1 and 1.618034.

(In order to check accuracy I try it with 10/9 places::

1.6180339887 and. 618033989, with rounding off on my calculator, so we have a continuing series.)

By this crude and curious method I have avoided engaging , in true classical Greek fashion, with the irrational square root of 5, which the algebraic solutions brings up. I suspect that this method can only be done with numbers, that it has no analog with stick or string which a Greek architechtural workman could have used.

A mathematical friend inspected this last method, and commented that I might point out to the general reader to the fact that the way. .618 is characterized in the exhaustions paragraph, stems from the first equation 1) above:

$(1-x)/x = x$

and write the left-hand side a $1/x - x/x$, so you get

$1/x - 1 = x$

or $1/x = x + 1$

This says that when you add 1 to the number you want (.618), you get the reciprocal of that number.

One way to home in on it, aside from the random approximations you mention, is as follows:

Start with any convenient number, e.g.. 5

Add 1 --- getting 1.5 in this case.

Form the reciprocal --- getting 1/1.5 or 0.667........

Add 1 --- getting 1.6

Form the reciprocal --- getting .625

Add 1

Form reciprocal..."

Soon you see convergence. You can start with any other number (between 0 and 1) in the place of. 5, and get the same. .618 ultimately.

THE APPROACH FOR GREEK ARCHITECTURE

Now we come to another approach, which I believe was the one the Greeks used. First let me set the stage with some background material which bears on my solution:

a) Plato had described in the Meno common knowledge about the squaring of the square, by constructing a larger square based on the hypotenuse of the original diagrammed square. He doubled the area, and neatly avoided having to deal with the square root of 2 by simply squaring it and returning it the realm of usable numbers. What he had been dealing with was of course 1.414213562....

b) With such interesting returns from the experiment with the square, a next natural trial might well be dealing with a rectangle with an adjacent side twice the length of its partner, hence a 1 : 2 rectangle. Now by Pythagorean theorem the hypotenuse will be the square root of 5 which the Greek cannot deal with, nor will it give an interesting return if handled like the 1 : 1 square. (A larger square based on this diagonal will have an area of 25, not consonant with the original rectangular area of 2, hence not interesting to a Greek. Dead-end in this direction.)

c) There is a note in Herodotus, speaking of the Egyptians and Egyptian mathematical knowledge.. H W Turnbull, the distinguished algebraist of the l940's, remarks in an essay in The Great Mathematicians, on Herodotus' passage:

"A certain obscure passage in Herodotus can, by the slightest literal emendation, be made to yield excellent sense. It would imply that the area of each triangular face of the Pyramid is equal to the square of the vertical height. and this accords well with the actual facts. If this be so, the ratios of height, slope and base can be expressed in terms of the Golden Section, of the radius of a circle to the side of the inscribed decagon. In short there was already a wealth of geometrical and arithmetical results treasured by the priests of Egypt, before the early Greek travelers became acquainted with mathematics..."

d) Herodotus also mentions that the Egyptians had a regular class of mathematical technicians, whom he calls "rope measurers", who were used not only to measure out linear distances for surveying, but to establish complex geometric figures. The convenient whole numbers associated with the Pythagorean theorem, 3, 4 and 5 and any multiples of these, were well known to the Egyptians as basic information. Considering the complexities involved in the dimensioning of the Pyramids, it is clear that the rope measurers were the standard way of converting mathematical data to actual, architectural measurements. I mention this as especially important in relation to the following: section:

THE NEW SOLUTION

The 1 : 2 rectangle and the Egyptian Rope Measurers

It has always been a problem to undertand how the Greek architect and his consruction workers managed to incorporate into the design of large-scale temples like the Parthenon the "irrational" measurements which the Golden Mean requires. The Greeks had no system for handling irrational numbers in a theoretical manner, let alone applying irrational measurements to an actual conctruction project. Extending the numbers of the GM proportion from one place to another on a building in the process of construction would seem to have been impossible.

But the proportions are clearly there in fact. So at this point I want to introduce a method, which I take to be an independent discovery on my part, and the key to the use of the GM ratio in large scale applications in architecture, for example in Iktinos' GM based designs for the Parthenon..

a) I construct a 1 : 2 rectangle of any size, depending on what scale I am working with.

b) I fix a non-elastic string or tape to the lower left hand corner of this rectangle, and run it around a point (a nail) at the upper right hand corner then draw it down to the lower right corner. This adds the short side of the rectangle (1) to the diagonal (sqrt5).

c) I then take my string, hold the ends together, and stretching it out double, I halve its length. This is now (sqrt 5+1)/2 or numerically 1.618....., the number have been seeking for comparison to one (1).

d) I can take this string/number and use it as short side of a new larger rectangle, and construct a new larger rectangular figure with the same proportions preserved.

e) But I may want to get smaller, that is find the inverse (1/x) of 1.618 (which is. 618), I can do this by the string method too. I draw my line from left lower to right upper corner, bring down the line to lower right corner, and folding that back along the diagonal, I mark that point, which represents the subtraction of one (1) from sqrt5. If I take that remaining length of my line from the start to the mark, and fold it double, I get. 618 or the inverse of 1.618 (1/x).

To us in a day of exact measurements with electronic drafting equipment, may seem inconvenient to make architectural measurements with a string, but recall that all measurements of objects in the real world are still made with linear devices. In the West we use a plastic tape of high strength with low expansion, before that it was a woven marked tape laced with copper wires, earlier the "chain" and "rod" which were standards set to a given length. Surveyors still use tapes, usually of steel with a calculation for drop under a specific tension, rather than available optical distance measuring equipment, since they are more accurate. The rolling pi-wheel is used on highways for initial measurement, but never replaces the tape. (It is only in recent years that we have adopted use of wave lengths from certain elements (cadmium) at a fixed temperature, as a standard of length, but that is in laboratory situations only for establishing standards.)

The Greeks as carpenters, masons and architects, used direct measurement, that is, measurements transferred from one block or board to another which was to be made identical as possible Their constant mention of "congruence" points to this simple matching of direct size, which "congrue" if they match up in all their dimensions. Although the Greek mathematical intellectuals had a tendency to mask their connections with the handicrafts and trades as functioning on a lower plane, architecture and manufacturing were everywhere present in their society. The problem in this case is bringing together the Greek mathematical knowledge of the geometers with the practical transfer of mathematical proportions to actual buildings.

In short, I believe the Greeks first explored the possibilities inherent in the rectangle of 1 : 2 ratio, and found that this satisfied in realizable dimensions the ideal proportion which Plato had discussed in his projection of the Divided Line.. The next step was devising a (string) method which would permit transfer of proportional measurements to real objects under construction.

Plato had said that a line so divided into two unequal segments so that the smaller bore the same relationship to the larger, and the larger to the whole line, would represent a special kind of proportional relationship with important properties. Euclid discussed this relationship in his book on proportional in geometric terms, naturally stopping short of identifying exact numbers, which would have been inconceivable with the primitive Greek numerical system. That this was a commonly understood and accepted ratio can be inferred from its extensive use in the work of the 5th century architect Iktinos, who designed the Parthenon with the Golden Mean ratio throughout..

The QUESTION : How could the Greeks, who had a very poor number system which used letters of the alphabet without a zero, who furthermore were confused by "irrationals", as numbers which they could not calculate arithmetically ----. how could they have determined with exactitude the numbers which are discussing, and used them in architectural design?

The ANSWER: They used the above methods, establishing a rectangle of size consonant with the work to be done, ran a string or copper wire around the points as I have described, and could thus transfer a Golden Section ratio dimension to a column, to the spacing between columns, to a metope, to a plan layout. With knowledge of the properties of the 1 : 2 rectangle, and a mechanical linear method of transfer of measurement, they were able to devote themselves to subtle elements of design. And this without having to construct a numerical interface they way we have done. Our way is easier for us, with fine calculations, CAD layout with lines of no dimension, and plotters printing out to scale. But for this we have had to provide a great deal of physical equipment, and a great deal of intellectual training and preparation for any operation we undertake.

The Greek were more direct, their architecture is amazingly subtle and persuasive, and I think part of their artistry comes from their use of complex mental processes, coupled with very direct and simple ways of transferring ideas into wood and stone structures.

There is so much written about the Golden Section or Mean proportion, that a bibliography would be impossible here. I note just a few of the older references which might not appear from a websearch or Amazon booksellers:

Jay Hambridge: The Elements of Dynamic Symmetry NY l926

Tobias Danzig The Bequest of the Greeks l955

W W Rouse Ball: Mathematical Recreations and Essays, 11 ed l940 or one of the later revisions.

William Harris
Prof. Em. Middlebury College
www.middlebury.edu/~harris